Lecture 1

Tuesday, September 1, 2020

<u>Something to ponder:</u>

* Why are the doctrines of the Gospel so easy to understand while the concepts of math, although less important, so difficult to understand?
* Learning things pertaining to God's creation is a commandment (D&C 88, verse 77–79).

<u>An overview of the course:</u>

$$\underbrace{\text{Ordinary}}_{\substack{\text{function of} \\ \text{one variable}}}\ \underbrace{\text{Differential Equation}}_{\substack{\text{a relation between a function} \\ \text{and its derivatives}}} \quad \text{(ODE)}$$

For example,

$$\left.\begin{array}{l} y' = t + y \\ yy' = t^2 \\ y'' + yy' = 1 \\ \ldots \end{array}\right\} \quad \text{ODEs}$$

In each equation, the unknown is y as a function of t. Note that the unknown is a function, not just a number.

Ordinary Diferential Equations is a particular form of Partial Differential Equations why the function can have more than one variable. An example of PDE is

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

where $u = u(x,t)$ is a function of x and t. The notations $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}, \ldots$ are called partial derivatives of u.

$\frac{\partial u}{\partial t}$ is the derivative of u w.r.t. t (while freezing x).

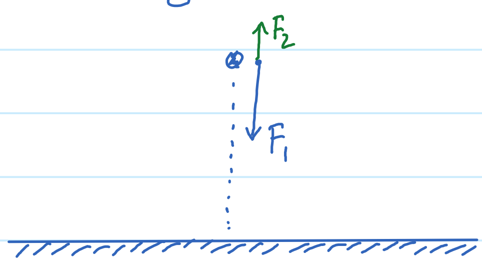
$$\frac{\partial u}{\partial t} = \lim_{h\to 0} \frac{u(x,t+h) - u(x,t)}{h}$$

Similarly,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,t) - u(x,t)}{h}$$

PDE are prevalent in real life. We will only consider ODE in this course. An understanding of ODE will help us better understand PDE. Already, ODE is quite prevalent in real life. They come, usually naturally, from natural phenomena. Let us consider a few examples

(1) The falling object due to gravity:



Consider an object in the picture. It is held at certain altitude from the ground and then released.
One is interested in the velocity of the object as a function of time: $v = v(t)$.

Before even solving for v, one can say that it should be an increasing function of t (until the object hits the ground). By Newton's Second Law of Motion:

$$F = ma$$

F: total force applied on the object

a: acceleration of the object

We know that $a = a(t) = v'(t) = v'$.
There are two forces applied on the object: the gravitional force F_1 and the air resistence force F_2. They compete each other.

$$F = \underbrace{F_1}_{=mg} - \underbrace{F_2}_{=?}$$

How to find F_2 is a complicated problem. It is the problem of computing drag force, which is an important and difficult problem is fluid dynamics However, one can make a simple assumption on F_2 by the following observation (by Newton): the faster the object moves, the stronger the air resistence. Newton postulated that F_2 is proportional to v. This assumption is often a good one provided that v is not too big.

$$F_2 = k v$$

constant of air resistance, can be found through experiments

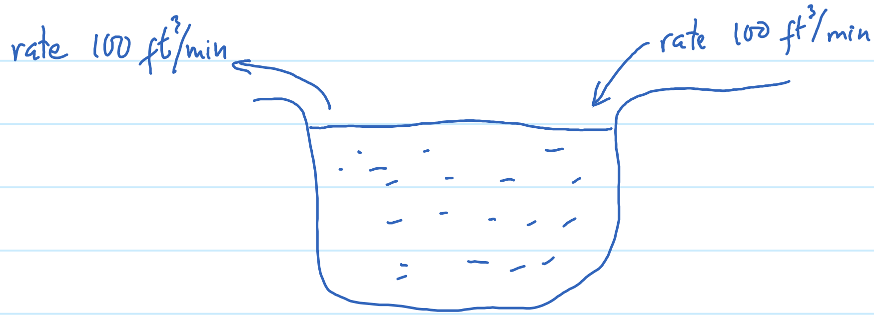
Therefore, we get an equation

$$v' = g - \frac{k}{m} v .$$ ODE

Here g, k, m are constant (w.r.t time) that can be found. The unknown is v as a function of time.

② Purifying the Great Salt Lake:

Suppose we want to purify the Great Salt Lake by pumping in fresh water and pumping out water in the lake at the same speed (so that the total volume of water in the lake is the same at all time). How well does this method work?



By a simple Google search, one can find that the volume of water in the lake is $V \approx 668 \times 10^9 \ ft^3$ and the salt content is $c_0 \approx 13\% = 0.13$. We are interested in the salt content is as a function of time.

$c = c(t)$: salt content (lbs/ft^3)

Let us call $y = y(t)$ the total amount of salt contained in the lake. Thus, $y = cV$ (lbs). By examining the rate of change of salt over time, we obtain the equation

$$\frac{dy}{dt} = \underbrace{0}_{\substack{\text{contribution} \\ \text{of the in-flow}}} - \underbrace{100 \times \frac{y}{V}}_{\substack{\text{rate of salt (lbs)} \\ \text{escaping the lake}}}$$

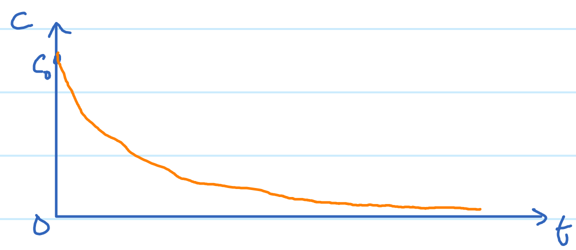
Thus,

$$\frac{dy}{dt} = -\frac{100}{V} y$$

or in terms of c:

$$c' = -\frac{100}{V} c \quad \leftarrow \text{ODE}$$

One can solve this equation by a method introduced later: $c(t) = c_0 e^{-\frac{100}{V}t}$. Because $V \gg 100$, $c(t)$ decreases slowly as t increases from 0, but then decreases exponentially to zero as t goes to infinity. This suggests that the method satisfies the demand although it will take a long time.



We see that two different phenomena share the same type of ODE:

$$y' = a - by$$

where a and b are constant. Understanding this ODE helps us understand not only a single phenomenon but also a class of phenomena that can be modeled by the same equation.

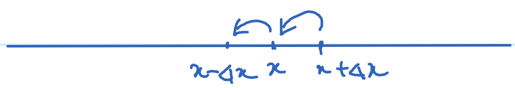
③ Heat transfer on a rod:

Suppose we have a rod so thin that one can draw it as a line segment.

We are interested in the temperature of the rod at each location at each time instance. The temperature is thus a function of x – the position, and t – time.

$$u = u(x,t).$$

Let us see how u changes with respect to time.



At position x, the change of temperature from time t to $t+\Delta t$ is caused by the difference of temperature between x, $x-\Delta x$ and $x+\Delta x$

$$\begin{aligned} u(x,t+\Delta t) - u(x,t) &= \big(u(x+\Delta x,t) - u(x,t)\big) - \big(u(x,t) - u(x-\Delta x,t)\big) \\ &= u(x+\Delta x,t) + u(x-\Delta x,t) - 2u(x,t) \end{aligned}$$

Thus,

$$\underbrace{\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}}_{\frac{\partial u}{\partial t}} = \underbrace{\frac{(\Delta x)^2}{\Delta t}}_{D \text{ (heat transfer coefficient)}} \underbrace{\frac{u(x+\Delta x,t) + u(x-\Delta x) - 2u(x,t)}{(\Delta x)^2}}_{\frac{\partial^2 u}{\partial x^2}}$$

Heat equation:

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}$$

This equation is also called the diffusion equation because it models a wide range of diffusive processes.